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Soliton solutions for the non-autonomous discrete-time Toda lattice equation

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Abstract

We construct N-soliton solution for the non-autonomous discrete-time Toda lattice equation, which is a generalization of the discrete-time Toda equation such that the lattice interval with respect to time is an arbitrary function in time.

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1. Introduction

In this paper, we consider the nonlinear partial difference equation given by

$$\begin{aligned} &A_n^{t+1} + B_n^{t+1} + \lambda_{t+1} = A_n^t + B_{n+1}^t + \lambda_t, \\ &A_{n-1}^{t+1} B_n^{t+1} = A_n^t B_n^t, \end{aligned} \qquad n \in \mathbb{Z}, \tag{1}$$

where n and t are the independent variables, A_n^t and B_n^t are the dependent variables and λ_t is an arbitrary function in t, respectively. In the physical context, the variables n, t, A_n^t and B_n^t correspond to the lattice site, the discrete time and the fields, respectively. Equation (1) is equivalent to the following equation:

$$\begin{aligned} &J_n^{t+1} - \delta_{t+1} V_{n-1}^{t+1} = J_n^t - \delta_t V_n^t, \\ &V_n^{t+1}\left(1 - \delta_{t+1} J_n^{t+1}\right) = V_n^t\left(1 - \delta_t J_{n+1}^t\right), \end{aligned} \qquad n \in \mathbb{Z}, \tag{2}$$

where the variables are related as

$$A_n^t = -\lambda_t + J_n^t, \qquad B_n^t = -\lambda_t^{-1} V_{n-1}^t, \qquad \delta_t = \lambda_t^{-1}, \tag{3}$$

respectively. When λ_t or δ_t is a constant, equation (1) or (2) reduces to the discrete-time Toda equation proposed by Hirota [2, 4]. Moreover, equation (2) yields the celebrated Toda lattice equation

$$\begin{aligned} &\frac{\mathrm{d}J_n}{\mathrm{d}t} = V_{n-1} - V_n, \\ &\frac{\mathrm{d}V_n}{\mathrm{d}t} = V_n(J_n - J_{n+1}), \end{aligned} \qquad n \in \mathbb{Z}, \tag{4}$$

in the continuous limit $\delta_t = \delta \to 0$.

Equation (1) was proposed by Spiridonov and Zhedanov in [17], where the equation is called just 'the discrete-time Toda lattice'. On the other hand, equation (2) was proposed by Hirota [5], and called 'the random-time Toda equation'. However, it appears that those names are not appropriate for equations (1) and (2), since the former name usually refers to the case where λ_t and δ_t are constants, and the latter is somewhat misleading. In this paper, we call equations (1) and (2) 'the non-autonomous discrete-time Toda lattice equation'. The non-autonomous discrete-time Toda lattice equation is written in the Lax form

$$L_{t+1}R_{t+1} + \lambda_{t+1} = R_t L_t + \lambda_t, \tag{5}$$

where L_t and R_t are difference operators defined by

$$L_t = A_n^t + \mathrm{e}^{-\partial_n}, \qquad R_t = B_{n+1}^t\,\mathrm{e}^{\partial_n} + 1, \tag{6}$$

respectively. The Lax equation (5) is the compatibility condition of the spectral problem equation

$$\Psi_n^{t+1} = R_t\Psi_n^t = B_{n+1}^t\Psi_{n+1}^t + \Psi_n^t, \qquad (x-\lambda_t)\Psi_n^t = L_t\Psi_n^{t+1} = A_n^t\Psi_n^{t+1} + \Psi_{n-1}^{t+1}, \tag{7}$$

where x is a spectral parameter and Ψ_n^t is a wavefunction.

An important feature of soliton equations, including the Toda lattice and the discrete-time Toda equations, is that they admit a wide class of exact solutions, such as soliton solutions. Moreover, these solutions are expressed by determinants or Pfaffians [6], which are regarded as a characteristic property of integrable systems according to the Sato theory [11]. It is known that the discrete-time Toda equation (when λ_t is a constant) admits two kinds of determinant solutions. One is the Hankel-type determinant solution, in which the lattice site n appears as the determinant size [3, 7]. Another one is the Casorati determinant solution which describes soliton-type solutions [4]. In this solution, the determinant size corresponds to the number of solitons. The Hankel-type determinant solution for the non-autonomous discrete-time Toda lattice equation was constructed in [12, 13]. The purpose of this paper is to present explicit N-soliton solutions for the non-autonomous discrete-time Toda lattice equation in the form of the Casorati determinant.

2. Soliton solution for the non-autonomous discrete-time Toda lattice equation

For any $N \in \mathbb{Z}_{>0}$, we first define $N\times N$ Casorati determinants τ_n^t and σ_n^t as

$$\tau_n^t = \begin{vmatrix} \varphi_1^t(n) & \varphi_1^t(n+1) & \cdots & \varphi_1^t(n+N-1) \\ \varphi_2^t(n) & \varphi_2^t(n+1) & \cdots & \varphi_2^t(n+N-1) \\ \vdots & \vdots & & \vdots \\ \varphi_N^t(n) & \varphi_N^t(n+1) & \cdots & \varphi_N^t(n+N-1) \end{vmatrix}, \tag{8}$$

$$\sigma_n^t = \begin{vmatrix} \psi_1^t(n) & \psi_1^t(n+1) & \cdots & \psi_1^t(n+N-1) \\ \psi_2^t(n) & \psi_2^t(n+1) & \cdots & \psi_2^t(n+N-1) \\ \vdots & \vdots & & \vdots \\ \psi_N^t(n) & \psi_N^t(n+1) & \cdots & \psi_N^t(n+N-1) \end{vmatrix}, \tag{9}$$

where the entries $\varphi_i^t(n)$ and $\psi_i^t(n)$ $(i = 1, \ldots, N)$ satisfy the linear relations

$$\varphi_i^{t+1}(n) = \varphi_i^t(n) - \mu_t \varphi_i^t(n+1), \tag{10}$$

$$\psi_i^t(n) = \varphi_i^{t-1}(n) - \mu_t \varphi_i^{t-1}(n+1), \tag{11}$$

$$P_i^t \varphi_i^{t-1}(n) = \psi_i^t(n) - \mu_t \psi_i^t(n-1), \tag{12}$$

with μ_t being an arbitrary function in t, and P_i^t given by

$$P_i^t = (1 - p_i \mu_t)\left(1 - p_i^{-1}\mu_t\right), \qquad i = 1, \ldots, N. \tag{13}$$

For $N = 0$, we put $\tau_n^t = \sigma_n^t = 1$. Then, the main result of this paper is given as follows:

Theorem 1. *For τ_n^t defined above, the functions*

$$A_n^t = -\mu_t^{-1} \frac{\tau_n^t \tau_{n+1}^{t+1}}{\tau_n^{t+1} \tau_{n+1}^t}, \qquad B_n^t = -\mu_t \frac{\tau_{n-1}^{t+1} \tau_{n+1}^t}{\tau_n^t \tau_n^{t+1}}, \qquad \lambda_t = \mu_t + \mu_t^{-1} \tag{14}$$

satisfy the non-autonomous discrete-time Toda lattice equation (1).

As was pointed out in [12, 13], the auxiliary τ function σ_n^t plays an essential role although it does not appear in the final result.

Proposition 2. *τ_n^t and σ_n^t satisfy the following bilinear difference equations:*

$$\tau_n^{t-1}\tau_n^{t+1} - \tau_n^t \sigma_n^t = \mu_{t-1}\mu_t\left(\tau_{n-1}^{t+1}\tau_{n+1}^{t-1} - \tau_n^t \sigma_n^t\right), \tag{15}$$

$$\mu_t \sigma_n^t \tau_{n+1}^t - \mu_{t-1}\tau_n^t \sigma_{n+1}^t = (\mu_t - \mu_{t-1})\tau_n^{t+1}\tau_{n+1}^{t-1}. \tag{16}$$

Theorem 1 is a direct consequence of proposition 2. Actually, multiplying equation (16) by $1 - (\mu_t \mu_{t-1})^{-1}$, we have

$$\left(\mu_t - \mu_{t-1}^{-1}\right)\sigma_n^t \tau_{n+1}^t - \left(\mu_{t-1} - \mu_t^{-1}\right)\tau_n^t \sigma_{n+1}^t = (\lambda_t - \lambda_{t-1})\tau_n^{t+1}\tau_{n+1}^{t-1}. \tag{17}$$

Multiplying equation (17) by $\tau_n^t \tau_{n+1}^t$ and using equation (15), we have

$$\left(\tau_{n+1}^t\right)^2\left(\mu_t \tau_{n-1}^{t+1}\tau_{n+1}^{t-1} - \mu_{t-1}^{-1}\tau_n^{t-1}\tau_n^{t+1}\right) - \left(\tau_n^t\right)^2\left(\mu_{t-1}\tau_{n-1}^{t+1}\tau_{n+1}^{t-1} - \mu_t^{-1}\tau_n^{t-1}\tau_n^{t+1}\right)$$
$$= (\lambda_t - \lambda_{t-1})\tau_n^t \tau_n^{t+1}\tau_{n+1}^{t-1}\tau_{n+1}^t. \tag{18}$$

Dividing equation (18) by $\tau_n^t \tau_n^{t+1}\tau_{n+1}^{t-1}\tau_{n+1}^t$, we obtain the first equation of equation (1). The second equation is an identity under the variable transformation (14).

Remark 3.

(i) If we choose the functions $\varphi_i^t(n)$ and $\psi_i^t(n)$ as exponential-type functions

$$\varphi_i^t(n) = \alpha_i p_i^n \prod_{j=t_0}^{t-1}(1 - p_i \mu_j) + \beta_i p_i^{-n} \prod_{j=t_0}^{t-1}\left(1 - p_i^{-1}\mu_j\right), \tag{19}$$

$$\psi_i^t(n) = \alpha_i p_i^n (1 - p_i \mu_t) \prod_{j=t_0}^{t-2}(1 - p_i \mu_j) + \beta_i p_i^{-n}\left(1 - p_i^{-1}\mu_t\right) \prod_{j=t_0}^{t-2}\left(1 - p_i^{-1}\mu_j\right), \tag{20}$$

respectively, where α_i, β_i and p_i $(i = 1, \ldots, N)$ are parameters, we have the N-soliton solution. As is shown in [6, 14], τ functions for soliton solutions are expressed as Casorati determinants whose entries are given by exponential-type functions.

(ii) In the case where μ_t is a constant, the bilinear equations (15) and (16) reduce to

$$\tau_n^{t-1}\tau_n^{t+1} - \left(\tau_n^t\right)^2 = \mu^2\left[\tau_{n-1}^{t+1}\tau_{n+1}^{t-1} - \left(\tau_n^t\right)^2\right], \tag{21}$$

which is the bilinear equation of the discrete-time Toda equation [2]. Indeed, the N-soliton solution also reduces to that for the discrete-time Toda equation.

(iii) The functions $\varphi_i^t(n)$ $(i = 1, \ldots, N)$ satisfy the spectral problem equation

$$\begin{aligned} &\varphi_i^{t+1}(n) = -\mu_t \varphi_i^t(n+1) + \varphi_i^t(n), \\ &(x_i - \lambda_t)\varphi_i^t(n) = -\mu_t^{-1}\varphi_i^{t+1}(n) + \varphi_i^{t+1}(n-1), \qquad x_i = p_i + p_i^{-1}. \end{aligned} \tag{22}$$

Equation (22) is the spectral problem equation (7) with $A_n^t = -\mu_t^{-1}$, $B_n^t = -\mu_t$ and $\lambda_t = \mu_t + \mu_t^{-1}$, which is the simplest solution for the non-autonomous discrete-time Toda lattice equation (1).

3. Proof of proposition 2

In this section we prove proposition 2 by using the technique developed in [14, 15]. The bilinear equations (15) and (16) reduce to the Plücker relations, which are quadratic identities among the determinants whose columns are properly shifted. Therefore, we first prepare such difference formulae that express shifted determinants in terms of τ_n^t or σ_n^t. For simplicity, we introduce the notation

$$\tau_n^t = |0_t \quad 1_t \quad \cdots \quad (N-1)_t|, \qquad \sigma_n^t = |\hat{0}_t \quad \hat{1}_t \quad \cdots \quad \widehat{(N-1)}_t|, \tag{23}$$

where the symbols k_t and $\hat{k}_t$ are column vectors given by

$$k_t = \begin{pmatrix} \varphi_1^t(n+k) \\ \varphi_2^t(n+k) \\ \vdots \\ \varphi_N^t(n+k) \end{pmatrix}, \qquad \hat{k}_t = \begin{pmatrix} \psi_1^t(n+k) \\ \psi_2^t(n+k) \\ \vdots \\ \psi_N^t(n+k) \end{pmatrix}, \tag{24}$$

respectively.

Lemma 4. *The following formulae hold:*

$$\tau_n^t = |0_t \quad 1_t \quad \cdots \quad (N-2)_t \quad (N-1)_t|, \tag{25}$$

$$\tau_n^{t-1} = |0_t \quad 1_t \quad \cdots \quad (N-2)_t \quad (N-1)_{t-1}|, \tag{26}$$

$$\mu_{t-1}\tau_n^{t-1} = |0_t \quad 1_t \quad \cdots \quad (N-2)_t \quad (N-2)_{t-1}|, \tag{27}$$

$$\left(\prod_{i=1}^N P_i^t\right)^{-1} \tau_n^{t+1} = |\tilde{0}_{t+1} \quad 1_t \quad \cdots \quad (N-2)_t \quad (N-1)_t|, \tag{28}$$

$$\left(\prod_{i=1}^N P_i^t\right)^{-1} \mu_t\tau_n^{t+1} = |\tilde{1}_{t+1} \quad 1_t \quad \cdots \quad (N-2)_t \quad (N-1)_t|, \tag{29}$$

$$(1-\mu_{t-1}\mu_t)\left(\prod_{i=1}^N P_i^t\right)^{-1} \sigma_n^t = |\tilde{0}_{t+1} \quad 1_t \quad \cdots \quad (N-2)_t \quad (N-1)_{t-1}|, \tag{30}$$

where the symbol $\tilde{k}_t$ is the column vector given by

$$\tilde{k}_t = \begin{pmatrix} \left(P_1^t\right)^{-1}\varphi_1^t(n+k) \\ \left(P_2^t\right)^{-1}\varphi_2^t(n+k) \\ \vdots \\ \left(P_N^t\right)^{-1}\varphi_N^t(n+k) \end{pmatrix}. \tag{31}$$

Proof of lemma 4. We first note that $\varphi_i^t(n)$ and $\psi_i^t(n)$ also satisfy the linear relations

$$\varphi_i^{t+1}(n) = \psi_i^t(n) - \mu_{t-1}\psi_i^t(n+1), \tag{32}$$

$$P_i^t\varphi_i^t(n) = \varphi_i^{t+1}(n) - \mu_t\varphi_i^{t+1}(n-1), \tag{33}$$

which follow from equations (10)–(12). Equation (25) is nothing but the definition. Equation (26) is derived as follows: subtracting $(j+1)$ th column multiplied by μ_{t-1} from jth column of τ_n^{t-1} for $j = 0, 1, \ldots, N-1$, and using equation (10), we have

$$\begin{aligned}\tau_n^{t-1} &= |0_{t-1} \quad 1_{t-1} \quad \cdots \quad (N-1)_{t-1}| \\ &= |0_{t-1} - \mu_{t-1}\times 1_{t-1} \quad 1_{t-1} \quad \cdots \quad (N-1)_{t-1}| \\ &= |0_t \quad 1_{t-1} \quad \cdots \quad (N-1)_{t-1}| \\ &= \cdots = |0_t \quad \cdots \quad (N-2)_t \quad (N-1)_{t-1}|,\end{aligned}$$

which is equation (26). Moreover, multiplying by μ_{t-1} the Nth column of the right-hand side of equation (26) and using equation (10), we have

$$\begin{aligned}\mu_{t-1}\tau_n^{t-1} &= |1_t \quad \cdots \quad (N-2)_t \quad \mu_{t-1}\times(N-1)_{t-1}| \\ &= |1_t \quad \cdots \quad (N-2)_t \quad (N-2)_t + \mu_{t-1}\times(N-1)_{t-1}| \\ &= |1_t \quad \cdots \quad (N-2)_t \quad (N-2)_{t-1}|,\end{aligned}$$

which is nothing but equation (27). Equations (28) and (29) can be proved in a similar manner by using equation (33). Equation (30) can be proved as follows: first note that σ_n^t is rewritten as

$$\sigma_n^t = |0_{t+1} \quad \cdots \quad (N-1)_{t+1} \quad \widehat{(N-1)}_t|, \tag{34}$$

which is shown in a similar manner by using equation (32). We also note that $\varphi_i^t(n)$ and $\psi_i^t(n)$ satisfy the relation

$$(1-\mu_t\mu_{t-1})\psi_i^t(n) = P_i^t\varphi_i^{t-1}(n) + \mu_t\varphi_i^{t+1}(n-1), \tag{35}$$

which can be derived by eliminating $\varphi_i^t(n-1)$ from equation (32) with n being replaced by $n-1$ and equation (12). Then, multiplying by $(1-\mu_t\mu_{t-1})$ the Nth column of the right-hand side of equation (34) and using equation (35), we obtain

$$\begin{aligned}&(1-\mu_t\mu_{t-1})\sigma_n^t \\ &\quad = |0_{t+1} \quad \cdots \quad (N-2)_{t+1} \quad (1-\mu_t\mu_{t-1})\times\widehat{(N-1)}_t| \\ &\quad = \begin{vmatrix} \varphi_1^{t+1}(n) & \cdots & \varphi_1^{t+1}(n+N-2) & P_1^t\varphi_1^{t-1}(n+N-1) \\ \varphi_2^{t+1}(n) & \cdots & \varphi_2^{t+1}(n+N-2) & P_2^t\varphi_2^{t-1}(n+N-1) \\ \vdots & & \vdots & \vdots \\ \varphi_N^{t+1}(n) & \cdots & \varphi_N^{t+1}(n+N-2) & P_N^t\varphi_N^{t-1}(n+N-1) \end{vmatrix} \\ &\quad = \cdots = \begin{vmatrix} \varphi_1^{t+1}(n) & P_1^t\varphi_1^t(n+1) & \cdots & P_1^t\varphi_1^t(n+N-2) & P_1^t\varphi_1^{t-1}(n+N-1) \\ \varphi_2^{t+1}(n) & P_2^t\varphi_2^t(n+1) & \cdots & P_2^t\varphi_1^t(n+N-2) & P_2^t\varphi_2^{t-1}(n+N-1) \\ \vdots & \vdots & \cdots & \vdots & \vdots \\ \varphi_N^{t+1}(n) & P_N^t\varphi_N^t(n+1) & \cdots & P_N^t\varphi_1^t(n+N-2) & P_N^t\varphi_N^{t-1}(n+N-1) \end{vmatrix} \\ &\quad = \prod_{i=1}^N P_i^t|\tilde{0}_{t+1} \quad 1_t \quad \cdots \quad (N-2)_t \quad (N-1)_{t-1}|,\end{aligned}$$

which is equation (30). This completes the proof of lemma 4. □

Now consider the following identity of the $2N \times 2N$ determinant:

$$\left|\begin{array}{c|ccc|ccc|cc} \tilde{0}_{t+1} & 0_t & \cdots & (N-2)_t & & \emptyset & & (N-1)_t & (N-1)_{t-1} \\ \hline \tilde{0}_{t+1} & & \emptyset & & 1_t & \cdots & (N-2)_t & (N-1)_t & (N-1)_{t-1} \end{array}\right| = 0. \tag{36}$$

Applying the Laplace expansion to the left-hand side of equation (36) and using lemma 4, we obtain

$$\begin{aligned} 0 &= |\tilde{0}_{t+1} \quad 0_t \quad 1_t \quad \cdots \quad (N-2)_t| \times |1_t \quad \cdots \quad (N-2)_t \quad (N-1)_t \quad (N-1)_{t-1}| \\ &\quad + |0_t \quad 1_t \quad \cdots \quad (N-2)_t \quad (N-1)_t| \times |\tilde{0}_{t+1} \quad 1_t \quad \cdots \quad (N-2)_t \quad (N-1)_{t-1}| \\ &\quad - |0_t \quad 1_t \quad \cdots \quad (N-2)_t \quad (N-1)_{t-1}| \times |\tilde{0}_{t+1} \quad 1_t \quad \cdots \quad (N-2)_t \quad (N-1)_t| \\ &= \mu_t \left(\prod_i^N P_i^t\right) \tau_{n-1}^{t+1} \times \mu_t \tau_{n+1}^{t-1} + \tau_n^t \times (1 - \mu_t \mu_{t-1}) \\ &\quad \times \left(\prod_i^N P_i^t\right) \sigma_n^t - \tau_n^{t-1} \times \left(\prod_i^N P_i^t\right) \tau_n^{t+1}, \end{aligned}$$

which is the bilinear equation (15).

The bilinear equation (16) can be proved by a similar technique. We prepare the following difference formulae.

Lemma 5. *The following formulae hold:*

$$\tau_n^t = |0_{t+1} \quad 1_{t+1} \quad \cdots \quad (N-2)_{t+1} \quad (N-1)_t|, \tag{37}$$

$$\mu_t \tau_{n+1}^t = |1_{t+1} \quad 2_{t+1} \quad \cdots \quad (N-1)_{t+1} \quad (N-1)_t|, \tag{38}$$

$$\sigma_n^t = |0_{t+1} \quad 1_{t+1} \quad \cdots \quad (N-2)_{t+1} \quad \widehat{(N-1)}_t|, \tag{39}$$

$$\mu_{t-1}\sigma_{n+1}^t = |1_{t+1} \quad \cdots \quad (N-2)_{t+1} \quad (N-1)_{t+1} \quad \widehat{(N-1)}_t|, \tag{40}$$

$$(\mu_{t-1} - \mu_t)\tau_{n+1}^{t-1} = |1_{t+1} \quad \cdots \quad (N-2)_{t+1} \quad (N-1)_t \quad \widehat{(N-1)}_t|. \tag{41}$$

Proof of lemma 5. Equations (37) and (38) are equivalent to equations (26) and (27), respectively. Equation (39) is the same as equation (34). Equation (40) can be derived by using equation (32) after multiplying by μ_{t-1} the Nth column of the right-hand side of equation (37). In order to prove equation (41), we note the following relation between $\varphi_i^t(n)$ and $\psi_i^t(n)$:

$$(\mu_{t-1} - \mu_t)\varphi_i^{t-1}(n) = \psi_i^t(n-1) - \varphi_i^t(n-1), \tag{42}$$

which can be obtained by eliminating $\varphi_i^{t-1}(n)$ from equation (10) with t being replaced by $t-1$ and equation (11). Multiplying by $\mu_{t-1} - \mu_t$ the Nth column of τ_{n+1}^{t-1} and using equation (42), we obtain equation (41). This completes the proof of lemma 5. □

The bilinear equation (16) is derived by applying the Laplace expansion to the left-hand side of the following identity:

$$\left|\begin{array}{ccc|ccc|ccc} 0_{t+1} & \cdots & (N-2)_{t+1} & & \emptyset & & (N-1)_{t+1} & (N-1)_t & \widehat{(N-1)}_t \\ \hline & \emptyset & & 1_{t+1} & \cdots & (N-2)_{t+1} & (N-1)_{t+1} & (N-1)_t & \widehat{(N-1)}_t \end{array}\right| = 0,$$

and using lemma 5. This completes the proof of proposition 2 and thus theorem 1.

4. Concluding remarks

In this paper we have presented the N-soliton solution for the non-autonomous discrete-time Toda lattice equation (1), which can be regarded as a generalization of the discrete-time Toda equation such that the lattice interval with respect to time is an arbitrary function in time.

Discrete soliton equations commonly arise as Bäcklund- or Darboux-type transformations for corresponding continuous soliton equations. In this context, a number of iterations of a Bäcklund transformation can be regarded as the discrete independent variable. The Bäcklund transformation admits one parameter, playing a role of the lattice interval, which can be an arbitrary function in the corresponding independent variable. In this sense, discrete soliton equations can be naturally extended to be non-autonomous (see, for example, [1, 16]). Also, such non-autonomous generalization can be mapped to autonomous case (the lattice intervals are constants) by certain gauge transformation [19]. However, it should be noted that such transformation does not map the soliton solutions to soliton solutions directly. It was recognized in [8, 9] that the discrete two-dimensional Toda lattice equation (or equivalently, the discrete KP equation) admits non-autonomous generalization keeping the determinantal structure of exact solutions.

It is known that various discrete soliton equations are derived from the discrete KP equation and its Bäcklund transformations. Therefore, it is expected that solutions of non-autonomous discrete soliton equations are discussed from this point of view. For example, the solutions of the non-autonomous discrete-time relativistic Toda equation have been constructed in this manner in [10].

However, direct reduction process from the non-autonomous discrete KP equation might not be sufficient. As we have shown in this paper, in the case of equation (2), clever introduction of an auxiliary τ function (σ_n^t in this paper) is critical, which does not appear in the autonomous or continuous cases. Careful investigation of this machinery may lead to various generalizations of discrete soliton equations and their solutions. This problem will be discussed in forthcoming papers.

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